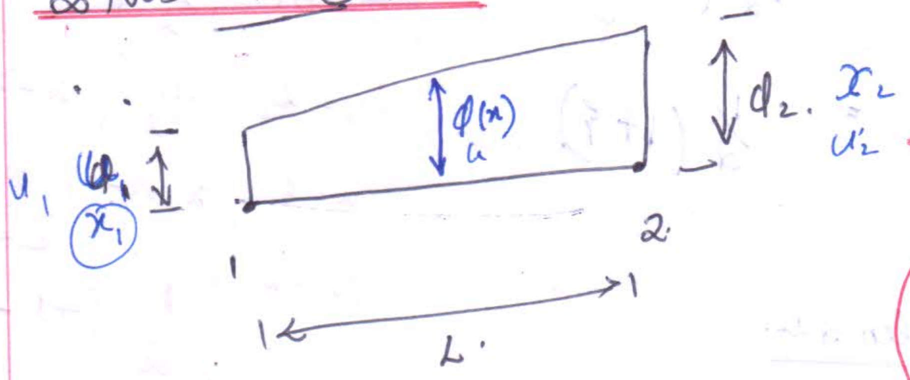
Lagrangian Shape functions.

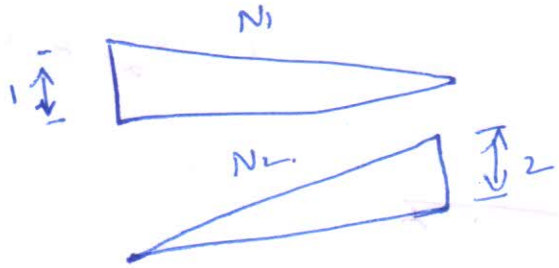
For 1D element:

2 Noded Element.



$$N_1 = \frac{l-x}{L}$$

$$N_2 = \frac{x}{L}$$



In FEM, it is a common practice to express the domain of an element in terms of natural Co-ordinates, ξ, η & ζ (xi, eta, Zeta)

$\xi \rightarrow$ xi, $\eta \rightarrow$ eta. $\zeta \rightarrow$ zeta)

The limits will vary between -1 & $+1$. Thus if the line element of fig win lies between the range of $(-1 \le \xi \le 1)$. then the nodes 1 and 2 are located at $\xi = -1$ & $\xi = 1$

With this modificat the Shape function in terms of ξ are written as.

$$N_1(\xi) = \frac{1}{2}(1-\xi)$$

$$N_2(\xi) = \frac{1}{2}(1+\xi)$$

We know

$N_1 = L_1 = \frac{1}{l}(x_2 - x)$

$N_2 = L_2 = \frac{1}{l}(x - x_1)$

$x = \xi$, $x_1 = (-1)$, $x_2 = 1$, $l = 2$ $(-1, +1)$

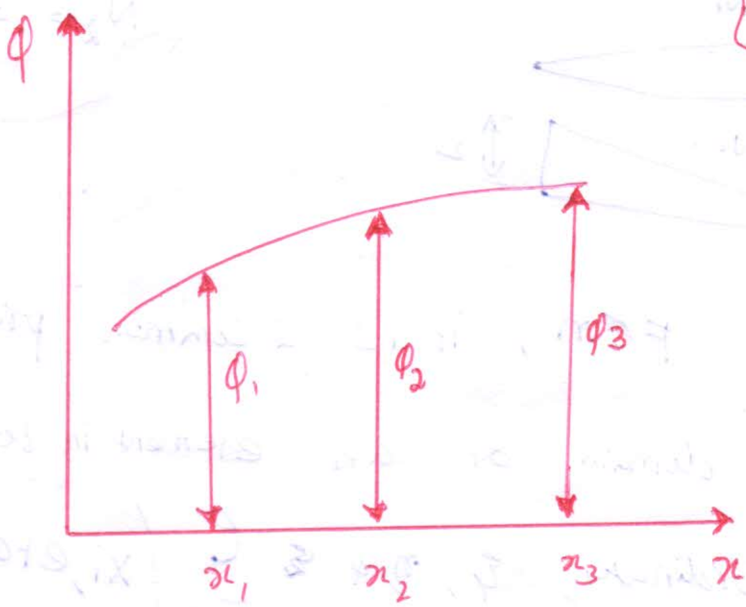
$N_1 = L_1 = \frac{1}{2}(+1 - \xi)$

$N_2 = L_2 = \frac{1}{2}(x - x_1)$

$= \frac{1}{2}(\xi - (-1))$

$= \frac{1}{2}(1 + \xi)$

three noded element.



$$\phi(x) = \alpha_1 + \alpha_2 x + \alpha_3 x^2$$

$$\phi(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{Bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{Bmatrix} = [P]\{x\}$$

$\phi = [C]\{\alpha\}$

The lagrangian functions are denoted by symbol L.. The shape functions has the unit value at node 1 $(x = x_1)$ and is zero at the other two nodes.

In this case it is easier to obtain shape function by intuition rather than solving the simultaneous equations.

To get zero value of L_1 at $x = x_2$, and $x = x_3$ is initially written as

$$L_1 = C(x - x_2)(x - x_3)$$

Writing unit value of function at $x = x_1$

$$1 = C(x_1 - x_2)(x_1 - x_3)$$

$$C = \frac{1}{(x_1 - x_2)(x_1 - x_3)}$$

hence

$$L_1 = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)}$$

L_2 at $x = x_1$ & $x = x_3$

$$L_2 = C(x - x_1)(x - x_3)$$

@ $x = x_2$

ve ...

$$1 = C(x_2 - x_1)(x_2 - x_3)$$

$$C = \frac{1}{(x_2 - x_1)(x_2 - x_3)}$$

$$L_2 = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)}$$

III

$$L_3 = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

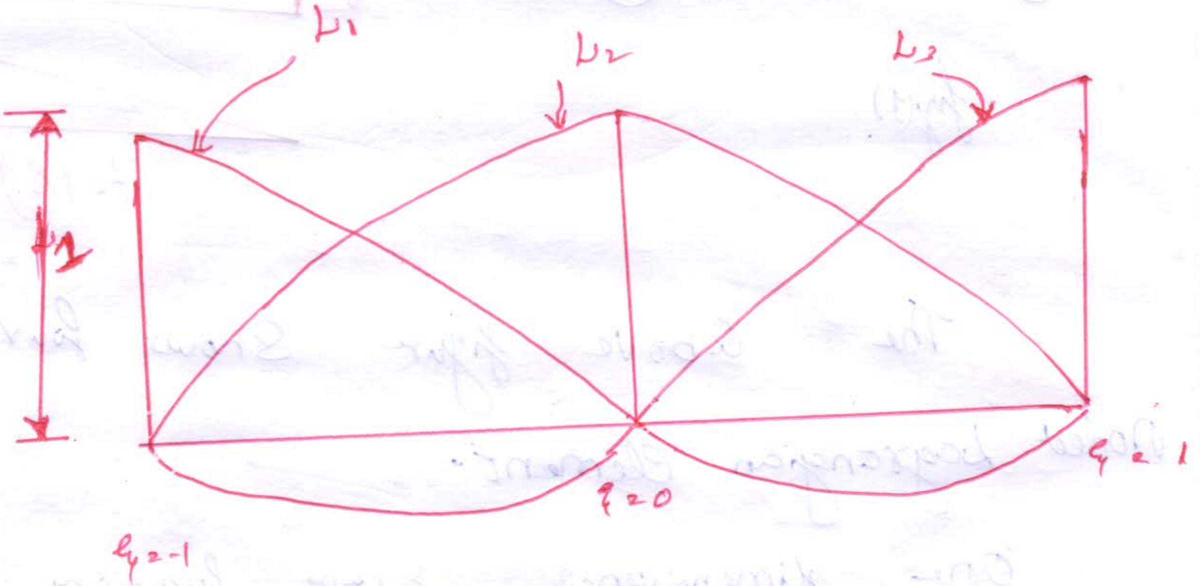
Lagrangian shape function for above three nodes are

$\xi_1 = -1$, $\xi_2 = 0$, $\xi_3 = 1$ sum in L_1, L_2 + L_3.

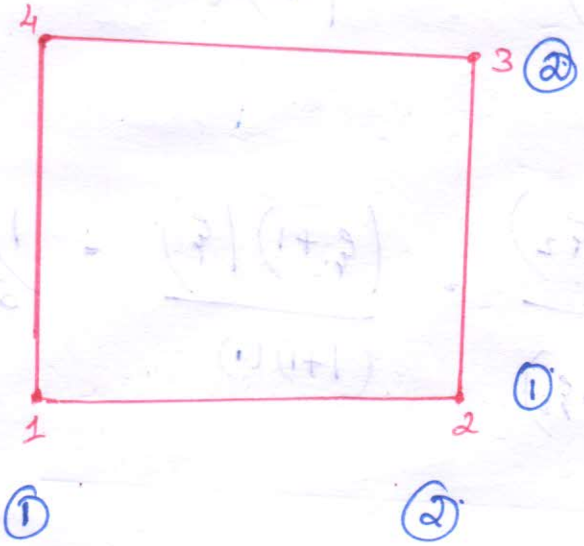
$$L_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = \frac{(\xi)(\xi - 1)}{(-1)(-1-1)} = \frac{1}{2}\xi(\xi - 1)$$

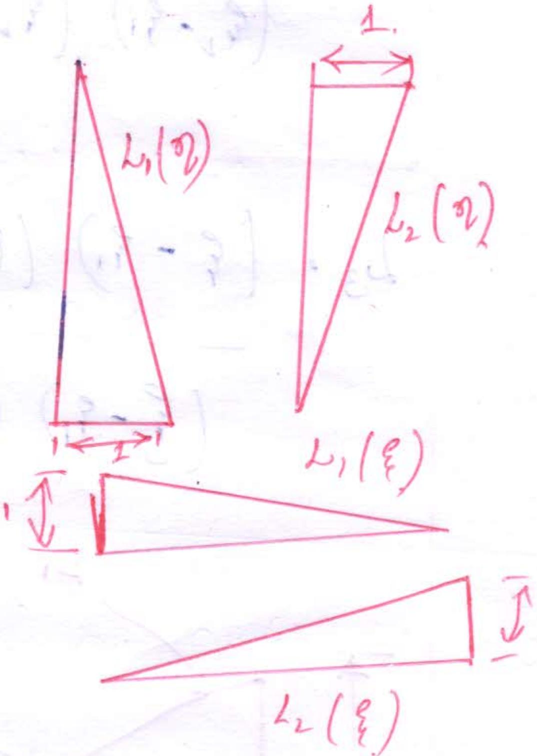
$$L_2 = \frac{(\xi-\xi_1)(\xi-\xi_3)}{(\xi_2-\xi_1)(\xi_2-\xi_3)} = \frac{(\xi+1)(\xi-1)}{(+1)(-1)} = (1-\xi)^2$$

$$L_3 = \frac{(\xi-\xi_1)(\xi-\xi_2)}{(\xi_3-\xi_1)(\xi_3-\xi_2)} = \frac{(\xi+1)(\xi)}{(1+1)(1)} = \frac{1}{2}\xi(\xi+1)$$



Four noded Lagrangian element :-





fig(1).

The above figure shows four noded Lagrangian element.

One dimensional shape functions are shown along the sides of the element.

$$L_1(\xi) = \frac{1}{2}(1-\xi) \quad , \quad L_1(\eta) = \frac{1}{2}(1-\eta)$$

$$L_2(\xi) = \frac{1}{2}(1+\xi) \quad , \quad L_2(\eta) = \frac{1}{2}(1-\eta)$$

Nodal shape functions are obtained by multiplying one dimension shape functions having unit value at concerned node of the element.

$$N_1 = L_1(\xi)\, L_1(\eta) = \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2 = L_2(\xi)\, L_1(\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = L_2(\xi)\, L_2(\eta) = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = L_1(\xi)\, L_2(\eta) = \frac{1}{4}(1-\xi)(1+\eta)$$

(or)

We know that

Lagrange interpolation function at node i is given by

$$f_i(\xi) = \frac{(\xi-\xi_1)(\xi-\xi_2)\cdots(\xi-\xi_{i-1})(\xi-\xi_{i+1})\cdots(\xi-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_n)}$$

$$= \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{(\xi-\xi_j)}{(\xi_i-\xi_j)}$$

(eg).

$$L_1(\xi) = \frac{(\xi-\xi_2)}{(\xi_1-\xi_2)} = \frac{(\xi-1)}{-1-(1)} = \frac{1}{2}(1-\xi)$$

$$L_2(\xi) = \frac{(\xi-\xi_1)}{(\xi_2-\xi_1)} = \frac{(\xi+1)}{1-(-1)} = \frac{1}{2}(1+\xi)$$

$$L_1(\eta) = \frac{(\eta-\eta_2)}{(\eta_1-\eta_2)} = \frac{1}{2}(1-\eta)$$

$$L_2(\eta) = \frac{(\eta-\eta_1)}{(\eta_2-\eta_1)} = \frac{1}{2}(1-\eta)$$

$$N_1^{(\xi,\eta)} = L_1(\xi)\,L_1(\eta) = \frac{(\xi-\xi_2)\,(\eta-\eta_2)}{(\xi_1-\xi_2)\,(\eta_1-\eta_2)} = \frac{(\xi-1)(\eta-1)}{-1-(1)\times-1-(1)} = \cancel{\tfrac{1}{4}(1-\xi)}$$

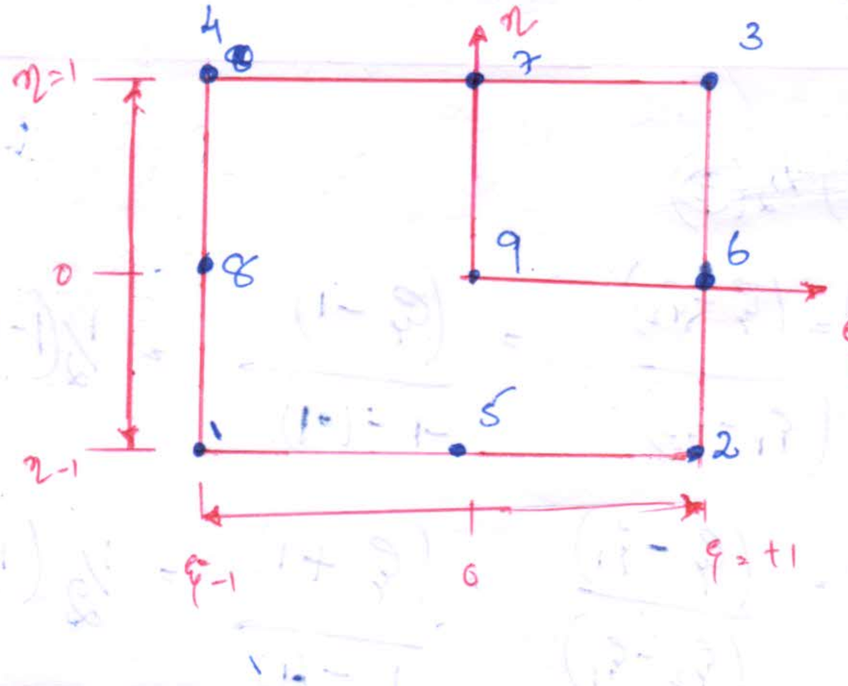
$$= \frac{1}{4}(1-\xi)(1-\eta)$$

$$N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

Nine Noded Element.



In the similar way, to the derivation of four node rectangular element, we can derive the shape function for the nine node rect. elem.

$$N_1 = L_1(\xi) L_1(\eta) = \frac{(\xi-\xi_2)(\xi-\xi_3)}{(\xi_1-\xi_2)(\xi_1-\xi_3)} \cdot \frac{(\eta-\eta_2)(\eta-\eta_3)}{(\eta_1-\eta_2)(\eta_1-\eta_3)}$$

$$\overset{(-1,-1)}{N_1} = \frac{1}{2}\xi(\xi-1)\,\frac{1}{2}\eta(\eta-1) = \frac{1}{4}\xi\eta(\xi-1)(\eta-1)$$

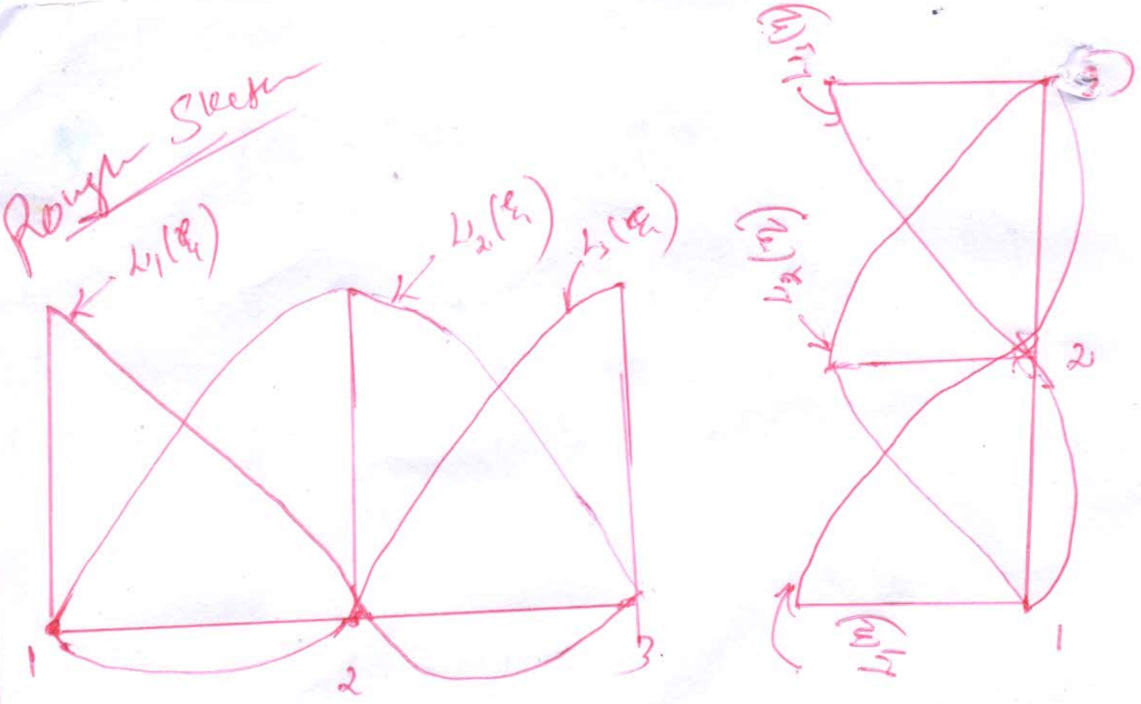
III^{ly}

$$N_2 = \frac{1}{4}\xi\eta(\xi+1)(\eta-1)$$

$$N_3 = \frac{1}{4}\xi\eta(\xi+1)(\eta+1)$$

$$N_4 = \frac{1}{4}\xi\eta(\xi-1)(\eta+1)$$

Rough Sketch



$$N_5 = L_2(\xi)L_1(\eta) = \frac{1}{2}\eta(1-\xi^2)(\eta-1)$$

$$N_6 = L_3(\xi)L_2(\eta) = \frac{1}{2}\xi(\xi+1)(1-\eta^2)$$

$$N_7 = L_2(\xi)L_3(\eta) = \frac{1}{2}\eta(1-\xi^2)(\eta+1)$$

$$N_8 = L_1(\xi)L_2(\eta) = \frac{1}{2}\xi(\xi-1)(1-\eta^2)$$

$$N_9 = L_2(\xi)L_2(\eta) = (1-\xi^2)(1-\eta^2)$$